THE GOLDEN SEQUENCE

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Abstract. This paper considers the sequence of fractional parts of multiples of the golden ratio. The main result characterizes the *Fibonacci* numbers by minimizing or maximizing this sequence.

1. Introduction and Preliminaries

Concerning the *Golden Section* there are two ratios – ratio 'big/small' and reciprocal ratio 'small/big' – which are calculated as follows

$$\Phi = \frac{\sqrt{5}+1}{2} = 1.618... \text{ and } \psi = \frac{\sqrt{5}-1}{2} = 0.618...$$

Obviously,

$$\Phi + \psi = \sqrt{5} \text{ and, basically, } \psi^2 = 1 - \psi$$

The *Fibonacci* numbers are recursively defined from $F_1 = 1$ and $F_2 = 1$ by

$$F_{n+2} = F_{n+1} + F_n \qquad (n = 1, 2, 3, ...)$$

In this paper, i, k, n always denote natural numbers ≥ 1. Most properties of Φ, ψ, and the *Fibonacci* sequence can be found in the well-known reference work [4]. In particular, Binet's formula

$$F_n = \frac{1}{\sqrt{5}}\big(\Phi^n - (-\psi)^n\big) \tag{1.1}$$

This formula allows both

- direct calculation of F_n from Φ and
- calculating the infinitesimal difference between $\frac{F_{n+1}}{F_n}$ and Φ

The latter means that $\frac{F_{n+1}}{F_n} - \Phi = \frac{(-\psi)^n}{F_n}$ and derives from:

$F_{n+1} - \Phi F_n \overset{(1.1)}{=} \frac{1}{\sqrt{5}}\big(- (-\psi)^{n+1} + \Phi(-\psi)^n\big) = (-\psi)^n \frac{1}{\sqrt{5}}(\psi + \Phi) = (-\psi)^n$. Moreover, this proves $F_{n+1} = \Phi F_n + (-\psi)^n$ and, hence,

$$\Phi F_n + (-\psi)^n \in \mathbb{N} \tag{1.2}$$

For $x \in \mathbb{R}$ let *floor* $\lfloor x \rfloor$ be the greatest integer $\leq x$ and *ceiling* $\lceil x \rceil = \lfloor x \rfloor + 1$. The *fractional part* $x - \lfloor x \rfloor$ is denoted by $\langle x \rangle$. Note that for $x, y, \delta \in \mathbb{R}$

$$\langle x + y \rangle = \begin{cases} \langle x \rangle + \langle y \rangle - 1 & \text{if } \langle x \rangle + \langle y \rangle \geq 1 \\ \langle x \rangle + \langle y \rangle & \text{if } \langle x \rangle + \langle y \rangle < 1 \end{cases} \tag{1.3}$$

and

$$x + \delta \in \mathbb{N} \text{ and } -1 < \delta < 1 \text{ implies } x + \delta = \begin{cases} \lceil x \rceil & \text{if } \delta > 0 \\ \lfloor x \rfloor & \text{if } \delta \leq 0 \end{cases} \tag{1.4}$$

These formulas will be useful later.

2. The Golden Sequence

The *Golden Sequence* is defined by the fractional parts of $n\Phi$

$$\langle \Phi \rangle, \langle 2\Phi \rangle, \langle 3\Phi \rangle, \ldots$$

Since $\Phi = 1 + \psi$, it holds $\langle n\Phi \rangle = \langle n\psi \rangle$ for all $n \geq 1$. Figure 1 illustrates the initial values of the *Golden Sequence*, using different symbols for $\langle n\Phi \rangle$ if n is a *Fibonacci* number.

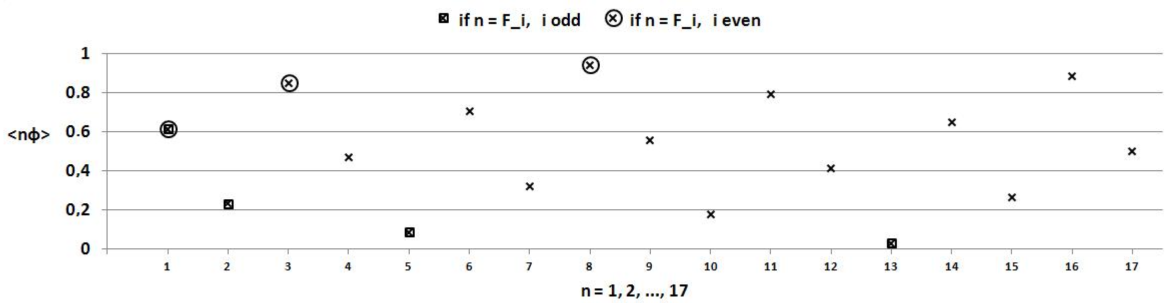

Figure 1. The *Golden Sequence* – *Fibonacci* elements emphasized

The subsequences $\langle F_1\Phi \rangle, \langle F_3\Phi \rangle, \langle F_5\Phi \rangle, \langle F_7\Phi \rangle, \ldots$ and $\langle F_2\Phi \rangle, \langle F_4\Phi \rangle, \langle F_6\Phi \rangle, \ldots$ are monotone. This results from

Lemma 2.1. *(cf.* [3], *p. 85, exercise 31) For all* $1 \leq k \in \mathbb{N}$

$\langle F_{2k-1}\Phi \rangle = \psi^{2k-1}$ *and* $\langle F_{2k}\Phi \rangle = 1 - \psi^{2k}$

Proof. By (1.2) and (1.4), $F_{2k-1}\Phi + (-\psi)^{2k-1} = \lfloor F_{2k-1}\Phi \rfloor$. Thus, $\langle F_{2k-1}\Phi \rangle = \psi^{2k-1}$. For the second part, again by (1.2) and (1.4), $F_{2k}\Phi + (-\psi)^{2k} = \lceil F_{2k}\Phi \rceil = \lfloor F_{2k}\Phi \rfloor + 1$. Thus, $\langle F_{2k}\Phi \rangle = 1 - \psi^{2k}$. □

Now, the *Fibonacci* elements – odd or even subscripts – of the *Golden Sequence* are shown to be extreme – minimal or maximal, respectively – until the next *Fibonacci* element but one.

Lemma 2.2. *For all* $1 \leq k \in \mathbb{N}$

(*a*) $\psi^{2k-1} \leq \langle i\Phi \rangle$ *for all* $1 \leq i < F_{2k+1}$ *and*
(*b*) $1 - \psi^{2k} \geq \langle i\Phi \rangle$ *for all* $1 \leq i < F_{2k+2}$

Proof. By 'interlaced' induction on k. (a) holds for $k = 1$ since $F_3 = 2$ and $\psi^1 \leq \langle \Phi \rangle$. (b) for $k = 1$ derives from $F_4 = 3$ and $1 - \psi^2 = \psi \geq \langle \Phi \rangle, \langle 2\Phi \rangle = 2\psi - 1 = 0.236...$

Suppose (a) and (b) hold for k.

(a) For $k + 1$, (a) follows since

$$\psi^{2k+1} < \langle i\Phi \rangle \text{ for all } F_{2k+1} < i < F_{2k+3} \tag{2.1}$$

For arbitrary i_0 between F_{2k+1} and F_{2k+3} there exists $j_0 < F_{2k+2}$ such that $i_0 = F_{2k+1} + j_0$.

By (b), $1-\psi^{2k} \geq \langle j_0\Phi\rangle$. By Lemma 2.1, $\langle F_{2k+1}\Phi\rangle = \psi^{2k+1}$. Thus, $\langle F_{2k+1}\Phi\rangle + \langle j_0\Phi\rangle \leq \psi^{2k+1} + 1 - \psi^{2k} < 1$. Equation (1.3) yields $\langle i_0\Phi\rangle = \langle F_{2k+1}\Phi + j_0\Phi\rangle = \langle F_{2k+1}\Phi\rangle + \langle j_0\Phi\rangle$. As $\langle j_0\Phi\rangle > 0$ it follows $\langle i_0\Phi\rangle > \langle F_{2k+1}\Phi\rangle = \psi^{2k+1}$ and (2.1) is proved.

(b) For $k+1$, (b) follows since

$$1-\psi^{2k+2} > \langle i\Phi\rangle \text{ for all } F_{2k+2} < i < F_{2k+4} \tag{2.2}$$

For arbitrary i_0 between F_{2k+2} and F_{2k+4} there exists $j_0 < F_{2k+3}$ such that $i_0 = F_{2k+2} + j_0$.

By (a) for $k+1$, $\psi^{2k+1} \leq \langle j_0\Phi\rangle$. Again by Lemma 2.1, $\langle F_{2k+2}\Phi\rangle = 1-\psi^{2k+2}$. Thus, $\langle F_{2k+2}\Phi\rangle + \langle j_0\Phi\rangle \geq 1-\psi^{2k+2} + \psi^{2k+1} > 1$. Equation (1.3) yields $\langle i_0\Phi\rangle = \langle F_{2k+2}\Phi + j_0\Phi\rangle = \langle F_{2k+2}\Phi\rangle + \langle j_0\Phi\rangle - 1$. As $\langle j_0\Phi\rangle - 1 < 0$ it follows $\langle i_0\Phi\rangle < \langle F_{2k+2}\Phi\rangle = 1-\psi^{2k+2}$ and (2.2) is proved. □

Theorem 2.3. *A positive integer n is a Fibonacci number if and only if*

(a) $\langle n\Phi\rangle < \langle i\Phi\rangle$ *for all* $1 \leq i < n$ *or* (b) $\langle n\Phi\rangle > \langle i\Phi\rangle$ *for all* $1 \leq i < n$

Moreover,

(a) *holds iff* $n = F_{2k-1}$ *for some* $k \geq 1$, *and*

(b) *holds iff* $n = F_{2k}$ *for some* $k \geq 1$

Proof. It suffices to show the 'moreover' parts. Assume (a) holds. Let k be maximal such that $F_{2k-1} \leq n$. Thus, $F_{2k+1} > n$. By Lemma 2.2, it follows $\psi^{2k-1} \leq \langle n\Phi\rangle$. Now, $F_{2k-1} = n$ will be shown by *reductio ad absurdum*. Suppose $F_{2k-1} < n$. By (a), $\langle n\Phi\rangle < \langle F_{2k-1}\Phi\rangle = \psi^{2k-1}$. This contradicts $\psi^{2k-1} \leq \langle n\Phi\rangle$.

Conversely, let $n = F_{2k-1}$ for some $k \geq 1$. (a) is true if $k=1$, since $n = F_1 = 1$, and, if $k > 1$, by Lemma 2.2, for all $1 \leq i < F_{2k-1}$ it holds $\langle i\Phi\rangle \geq \psi^{2k-3} > \psi^{2k-1} = \langle F_{2k-1}\Phi\rangle = \langle n\Phi\rangle$.

Suppose (b) holds. Let k be maximal such that $F_{2k} \leq n$. Thus, $F_{2k+2} > n$. By Lemma 2.2, it follows $1-\psi^{2k} \geq \langle n\Phi\rangle$. Again, $F_{2k} = n$ will be shown by *reductio ad absurdum*. Suppose $F_{2k} < n$. By (b), $\langle n\Phi\rangle > \langle F_{2k}\Phi\rangle = 1-\psi^{2k}$. This contradicts $1-\psi^{2k} \geq \langle n\Phi\rangle$.

Conversely, let $n = F_{2k}$ for some $k \geq 1$. (b) is true if $k=1$, since $n = F_2 = 1$, and, if $k > 1$, by Lemma 2.2, for all $1 \leq i < F_{2k}$ it holds $\langle i\Phi\rangle \leq 1-\psi^{2k-2} < 1-\psi^{2k} = \langle F_{2k}\Phi\rangle = \langle n\Phi\rangle$. □

Characterizing conditions, other than that of Theorem 2.3, are stated in [1], i.e., a positive integer n is a *Fibonacci* number if and only if $5n^2-4$ or $5n^2+4$ is a complete square. This is a special case of a more general result from [2].

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References

[1] I. M. Gessel, *Problem H-187: n is a Fibonacci number if and only if $5n^2+4$ or $5n^2-4$ is a square*, The Fibonacci Quarterly **10** (1972), 417.

[2] C. Kimberling, *The equation $m^2-4k=5n^2$ and Unique Representations of Positive Integers*, The Fibonacci Quarterly **45** (2007), 304 – 312.

[3] D. E. Knuth, *The Art of Computer Programming, Vol. 1, Fundamental Algorithms*, 2nd edition, Addison Wesley, 1973.

[4] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section*, Ellis Horwood Limited, Chichester, 1989.

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